MATHEMATICS

STRUCTURE SPACES OF A VECTOR LATTICE AND ITS DEDEKIND COMPLETION

BY

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Introduction

Let E be an Archimedean vector lattice. A subset S of E is *closed* (order closed) if $\{a_\alpha\} \subset S$ and sup (a_α) in E implies sup (a_α) is an element of S. Given any subset S of E let $S^\perp = \{y \in E: |y| \wedge |x| = 0$ for all x in $S\}$. $S^\perp$ is a closed ideal in E and $S^\perp \cap S \subset \{0\}$. We will say that E has the *Riesz property* if every closed ideal I in E is complemented ie., if $E = I \oplus I^\perp$. F. Riesz showed that every Dedekind complete vector lattice possesses this very valuable property.

In [1], D. G. JOHNSON and J. E. KIST gave several necessary and sufficient conditions for E to have the Riesz property; among them that every structure space of E be extremally disconnected. This indicates a close relation between such vector lattices and Dedekind complete vector lattices. It is this relationship that we develop here. Our principal tool will be the Dedekind completion $\hat{E}$ of the Archimedean vector lattice E. The Riesz property suffices to define a mapping between any structure space of E and a suitably chosen structure space of $\hat{E}$. It is shown that for a certain class of structure spaces, this mapping is a homeomorphism. Necessary and sufficient conditions are then given for the extension of this homeomorphism to all structure spaces of E.

In the last section, we examine discrete vector lattices which, while not necessarily satisfying the Riesz property, still exhibit some of the relationships discussed above.

1. *Preliminaries*

Let E be a vector lattice. An ideal $P \subset E$ is prime if $x \wedge y$ in P implies that x is in P or that y is in P. Equivalently, P is a prime ideal if $x \wedge y = 0$ implies that either x or y is an element of P, or if $P \supset I \cap J$ where I and J are ideals in E implies that $P \supset I$ or $P \supset J$ (cf., [2]).

Let $\mathfrak{P}$ denote a collection of prime ideals of E. For a subset $\mathfrak{B}$ of $\mathfrak{P}$, the *kernel* of $\mathfrak{B}$ is the ideal $k(\mathfrak{B}) = \cap \{P: P \in \mathfrak{B}\}$; for an ideal I contained

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in E, the *hull* of I is the subset $h(I)=\{P: P \in \mathfrak{P}$ and $P \supset I\}$. If we define for $\mathfrak{B} \subset \mathfrak{P}$

$$\overline{\mathfrak{B}}=h(k(\mathfrak{B})), \text{ then } \mathfrak{B} \rightarrow \overline{\mathfrak{B}}$$

defines a closure operation, and hence a topology on $\mathfrak{P}$ (called the *hull-kernel* topology). If $k(\mathfrak{P})=\{0\}$ then the resulting topological space is called a *structure space* of E. $\mathfrak{P}$ will represent the structure space of all prime ideals in E.

For each element a in E, let $\mathfrak{P}_a=\{P \in \mathfrak{P}: a \notin P\}$. Then, the totality of sets $\mathfrak{P}_a$, for a in E, forms a base for the open sets in the *hull-kernel* topology.

Remark. The value of the structure space $\mathfrak{P}$ lies in the following fact: any Archimedean vector lattice E is isomorphic to a vector lattice of extended real-valued functions on $\mathfrak{P}$. Thus, structure spaces give rise to representations of the vector lattice. In fact, Johnson and Kist have shown [2] that a number of well-known representation theorems can be developed within the framework of prime ideals.

Let E be an Archimedean vector lattice. Then, there exists a Dedekind complete vector lattice $\hat{E}$ with the following properties:

(1) E is isomorphic to a subvector lattice of $\hat{E}$.

(2) For u in $\hat{E}$, $u=\sup\{a \in E: a \leqq u\}=\inf\{b \in E: b \geqq u\}$.

$\hat{E}$ is called the *Dedekind completion* of E. $\hat{E}$ is determined uniquely by (1) and (2) up to isomorphism. Nakano has shown [5] that having a Dedekind completion characterizes Archimedean vector lattices.

Examples might help to bring the various concepts introduced into sharper focus.

Examples. If S is the set of all equivalence classes of simple functions (finite linear combinations of characteristic functions of measurable sets) in the totally σ-finite measure space (X, μ), then S is an Archimedean vector lattice with the Riesz property which is not Dedekind complete. Its Dedekind completion is the space $L^{\infty}(X, \mu)$ of equivalence classes of bounded μ-measurable functions on X.

As a special case of the above example, we mention the space s of sequences of real numbers with finite range. Its Dedekind completion is the space of bounded sequences of real numbers. We refer to this example later in the paper.

We sketch a proof that S satisfies the conditions mentioned above. S is naturally embedded in $L^{\infty}(X, \mu)$ as a vector lattice. The boundedness of elements in $L^{\infty}(X, \mu)$ and the density of simple functions in this space readily imply condition (2) for Dedekind completion. So, $\hat{S}=L^{\infty}(X, \mu)$.

We show now that S has the Riesz property. Let $I \subset S$ be a closed ideal. Let 1 be the unit constant function on X and set $g=\sup\{f \wedge 1;$

$f \in I, f \geqq 0\}$ (supremum in $L^\infty(X, \mu)$). Then g is the characteristic function of some measurable set $M \subset X$ and hence is an element of S. For if not, there is a set A of positive measure and $0<\beta<\alpha<1$ such that $\beta \leqq g(x) \leqq \alpha$ for almost all $x \in A$. It is then possible to find a positive $f \in I$ and a set B contained in A of positive measure such that $f \wedge 1 > g$ on B, contradicting the definition of g. Now, since g is in S and I is closed, g is in I. If f is any positive element of S and $f|M'=0$ ($f|M'$ is the restriction of f to the complement of M), there is a scalar c such that $0<cf \leqq g$ and hence, f is in I. Conversely, if f is in I, $f|M'=0$. So, $I=\{f \in S: f|M'=0\}$. Let g' be the characteristic function of M'. For any positive h in S, then, let $h_1=hg$ and $h_2=hg'$. $h=h_1+h_2$, h_1 and h_2 are elements of S, h_1 is in I and $h_2 \wedge f=0$ for all positive f in I. So, $S=I \oplus I^\perp$.

2. *The mapping of the Structure Spaces of E and $\hat{E}$.*

We will assume throughout this section that E is an Archimedean vector lattice.

For a given ideal $I \subset E$, define $\hat{I}=\{u \in \hat{E}$: there is $x \in I$ for which $|u| \leqq x\}$. $\hat{I}$ is an ideal in $\hat{E}$ and $\hat{I} \cap E=I$. However, P a prime ideal in E does not imply that $\hat{P}$ is a prime ideal in $\hat{E}$. (For such an example, refer to section 3.) We do have, however, the following theorem.

THEOREM 2.1. *If E has the Riesz property and P is a prime ideal in E, then $\hat{P}$ is a prime ideal in $\hat{E}$.*

PROOF. Let u and v be elements of $\hat{E}$ such that $u \wedge v=0$. Suppose there is an x in E such that $0 \leqq x \leqq u$ but $x \notin P$. Then, for all y in E with $0 \leqq y \leqq v$, we have $x \wedge y=0$, hence $y \in P$. So, either $A=\{x \in E: 0 \leqq x \leqq \leqq u\} \subset P$ or $B=\{y: 0 \leqq y \leqq v\} \subset P$. Note that $B \subset A^\perp$ ($A^\perp$ in E) and $A \subset A^{\perp\perp}$. Now, $A^\perp \cap A^{\perp\perp}=\{0\}$. So, either $A^\perp \subset P$ or $A^{\perp\perp} \subset P$.

Suppose $A^\perp \subset P$. Let $y \in E$ be such that $v \leqq y$. Let $y'=y_{A^\perp}$ (the projection of y on $A^\perp$) and $y''=y_{A^{\perp\perp}}$. Then, $y=y'+y''$ since E satisfies the Riesz property. But $B \subset A^\perp$ implies that $y'' \wedge \bar{y}=0$ for all $\bar{y}$ in B, hence $y'' \wedge v=0$. So, $v \leqq y'$ which implies that $v \in \hat{P}$.

If, on the other hand, $A^{\perp\perp} \subset P$, choose x in E such that $u \leqq x$. Putting $x'=x_{A^\perp}$ and $x''=x_{A^{\perp\perp}}$ we obtain in a similar fashion that $u \leqq x''$ and hence $u \in \hat{P}$.

Thus either $u \in \hat{P}$ or $v \in \hat{P}$ which implies that $\hat{P}$ is a prime ideal in $\hat{E}$.

For an ideal $I \subset E$, define $f(I)=\hat{I}$. Then, f is a one-to-one mapping of the class of all ideals in E *into* the class of all ideals in $\hat{E}$. By the above theorem, when E satisfies the Riesz property, f takes prime ideals into prime ideals. If $\mathfrak{V}$ is a collection of prime ideals in E, and $k(\mathfrak{V})=\{0\}$ it is clear that $k(\mathfrak{W})=\{0\}$ where $\mathfrak{W}=f(\mathfrak{V})$ is the corresponding structure space in $\hat{E}$. So, f takes structure spaces of E into structure spaces of $\hat{E}$.

Examples s and S, given earlier in the paper, may both be used to show that not all ideals in $\hat{E}$ are of the form $\hat{I}$ for some ideal $I \subset E$; consequently, f does not map onto.

From the facts that $E \subset \hat{E}$ and E has the Riesz property it is easily shown that f maps a basis for $\mathfrak{V}$ (the one defined in section 1) into open sets in $\mathfrak{W}$. f is then a one-to-one open mapping of $\mathfrak{V}$ onto $\mathfrak{W}$. That f is not, in general, a homeomorphism will be seen later.

A prime ideal $P \supset I$ is a *minimal prime ideal belonging to the ideal* I if whenever Q is a prime ideal containing I, then $Q \not\subset P$. A minimal prime ideal belonging to the ideal $\{0\}$ is called simply a *minimal prime ideal.* The following two results from [2] will be useful:

(A) Every ideal in E is the intersection of all minimal prime ideals belonging to it.

(B) A prime ideal P in E is a minimal prime ideal belonging to the ideal I if and only if whenever $x \in P$, there is an element $y \notin P$ such that $|x| \wedge |y| \in I$.

(So, P is a minimal prime ideal if and only if whenever $x \in P$, there is a $y \notin P$ such that $|x| \wedge |y| = 0$).

THEOREM 2.2. *If E has the Riesz property and $P \subset E$ is a minimal prime ideal belonging to the ideal I, then $\hat{P}$ is a minimal prime ideal belonging to the ideal $\hat{I}$.*

PROOF. Let $u \in \hat{P}$, and $u \geqq 0$. Choose x in P such that $x \geqq u$. Then, there is $y \in E$, $y \notin P$ such that $x \wedge |y| \in I$ since P is minimal. So, $0 \leqq u \wedge \wedge |y| \leqq x \wedge |y|$ which implies that $u \wedge |y| \in \hat{I}$ but $y \notin \hat{P}$.

In particular then, if $\mathfrak{V}$ is a structure space of minimal prime ideals in E, $\mathfrak{W} = f(\mathfrak{V})$ is a structure space of minimal prime ideals in $\hat{E}$.

THEOREM 2.3. *Let E have the Riesz property. Then, if $\mathfrak{V}$ is a structure space of E consisting entirely of minimal prime ideals, the mapping f is a homeomorphism of $\mathfrak{V}$ and $\mathfrak{W}$.*

PROOF. Let $\mathfrak{W}_u$ be a basic open set in $\mathfrak{W}$. Then, $f^{-1}(\mathfrak{W}_u) = \{P \in \mathfrak{V}: u \notin \hat{P}\}$. Let $y \in E$, $0 \leqq y \leqq u$. Then, $\mathfrak{V}_y = \{P \in V : y \notin P\}$ is open in $\mathfrak{V}$. Let $\mathfrak{G} = \cup \{\mathfrak{V}_y : y \in E,\ 0 \leqq y \leqq u\}$. Then, $\mathfrak{G}$ is open in $\mathfrak{V}$ and $\mathfrak{G} \subset f^{-1}(\mathfrak{W}_u)$. We will show that $f^{-1}(\mathfrak{W}_u) = cl_{\mathfrak{V}}\mathfrak{G}$ which will imply that $f^{-1}(\mathfrak{W}_u)$ is open in view of the extremal disconnectivity of $\mathfrak{V}$. (See Introduction)

Let $\mathfrak{F}_z = \{P \in \mathfrak{V} : z \in P\} (z \in E^+)$ be a basic closed set in $\mathfrak{V}$ such that $\mathfrak{G} \subset \mathfrak{F}_z$. Then, $\mathfrak{V}_y \subset \mathfrak{F}_z$ for $0 \leqq y \leqq u$ and hence $y \wedge z = 0$ for all such y. For, if $y \wedge z > 0$, there is a prime ideal $P \in \mathfrak{V}$ such that $y \notin P$ and $z \notin P$ contradicting $\mathfrak{V}_y \subset \mathfrak{F}_z$. So, $u \wedge z = 0$ and hence $f^{-1}(\mathfrak{W}_u) \subset \mathfrak{F}_z$. We have then $\mathfrak{G} \subset f^{-1}(\mathfrak{W}_u) \subset cl_{\mathfrak{V}}\mathfrak{G}$. We need only show, then, that $f^{-1}(\mathfrak{W}_u)$ is closed in $\mathfrak{V}$.

Let $\mathfrak{F}$ be the complement of $f^{-1}(\mathfrak{W}_u)$. We will show that $\mathfrak{F}$ is open. $\mathfrak{F} = \{P \in \mathfrak{V} : u \in \hat{P}\}$. Let $P_0 \in \mathfrak{F}$. Then, u is in $\hat{P}_0$. So, there is an x in P_0 such that $x \geqq u$. Since P_0 is minimal, there is $z \in E$ such that $z \wedge x = 0$ and $z \notin P_0$ (see result (B)). Now, $z \notin P_0$ implies that $P_0 \in \mathfrak{V}_z$. Choose an arbitrary P in $\mathfrak{V}_z$. Then, $z \notin P$ but $z \wedge x = 0$ and hence $x \in P$ which implies that $u \in \hat{P}$. So, $P \in \mathfrak{F}$. Consequently, $P_0 \in \mathfrak{V}_z \subset \mathfrak{F}$ which implies that $\mathfrak{F}$ is open. This completes the proof.

Remark. The minimality of the prime ideals was used only to show that $f^{-1}(\mathfrak{W}_u)$ was closed in $\mathfrak{V}$. An implication of this, since f is a closed mapping, is that all sets $\mathfrak{W}_v$ for v in $\hat{E}$ are closed as well as open. The following theorem will show that this is true when and only when $\mathfrak{V}$ consists entirely of minimal prime ideals. The above proof, then, cannot be used to extend the theorem beyond spaces of minimal prime ideals.

THEOREM 2.4. *Let $\mathfrak{V}$ be a structure space for the vector lattice E. Then, all sets $\mathfrak{V}_x$ are closed if and only if every ideal in $\mathfrak{V}$ is minimal.*

PROOF. The sufficiency is proved in [2]. Assume all sets $\mathfrak{V}_x$ are closed. Let $P_0 \in \mathfrak{V}$ and choose a positive x in P_0. (We may assume $P_0 \neq \{0\}$ since in that case it is certainly minimal.) Let $\mathfrak{F} = \{P \in V : x \notin P\}$. $P_0 \notin \mathfrak{F}$. So, there is a basic open set $\mathfrak{V}_y$ containing P_0 and not meeting $\mathfrak{F}$. Hence, $y \notin P_0$ and since $\mathfrak{F} \cap \mathfrak{V}_y = \emptyset$, we have $x \wedge y = 0$. This implies that P_0 is a minimal prime ideal.

The general question as to how far this homeomorphism may be extended is not settled. The following theorems, however, give useful results in this direction.

1) f is generally not a homeomorphism of every structure space $\mathfrak{V}$ and its correspondent $\mathfrak{W}$ when E is not complete, even if E has the Riesz Property.

2) Yet, the homeomorphism of every pair $\mathfrak{V}$, $\mathfrak{W}$ is not enough to imply that E is complete (i.e., that E and $\hat{E}$ are isomorphic).

THEOREM 2.5. *Let E have the Riesz property. Then, the following are equivalent:*

(1) *f is a homeomorphism of all pairs $\mathfrak{V}$, $\mathfrak{W}$, where $\mathfrak{W} = f(\mathfrak{V})$.*

(2) *For every u in $\hat{E}^+$, there is $x \in E^+$ such that $u \leqslant x$ and for each prime ideal P, if $u \in \hat{P}$, then x is in P.*

(3) *For $u \in \hat{E}^+$, there is $x \in E$ such that $u \leqq x$, and for all $y \in E$ such that $u \leqq y \leqq x$, y and x generate the same principal ideal in E.*

PROOF. (1) $\Rightarrow$ (2). Consider the homeomorphism of the pair $\mathfrak{P}$, $\mathfrak{Q}$ ($\mathfrak{P}$ is the space of all prime ideals in E; $\mathfrak{Q}$ is its correspondent under f). Let u be a positive element of $\hat{E}$. Let $\mathfrak{F} = \{\hat{P} : u \in \hat{P}\}$. Then, if $\mathfrak{H} = f^{-1}(\mathfrak{F})$, $\mathfrak{H}$ is closed by assumption, that is to say, $h(k(\mathfrak{H})) = \mathfrak{H}$. Let $I = k(\mathfrak{H})$.

We will show that u is an element of $\hat{I}$. Suppose not. Then, by (A), there is a minimal prime ideal $\hat{Q}$ belonging to the ideal $\hat{I}$ such that $u \notin \hat{Q}$. (A *minimal* prime ideal $\tilde{Q}$ in $\hat{E}$ belonging to an ideal of the form $\hat{I}$ must be a $\hat{Q}$ for some ideal Q in E. For $\hat{I} \subset \tilde{Q}$ implies $\hat{I} \subset \hat{Q} \subset \tilde{Q}$ where $Q = \tilde{Q} \cap E$.) Moreover, $I \subset Q$. So, Q is a prime ideal in $h(k(\mathfrak{H}))$ but Q is not in $\mathfrak{H}$, a contradiction. So, $u \in \hat{I}$.

Now $u \in \hat{I}$ implies there is $x \in I$ such that $u \leqq x$. It follows then, that $\{\hat{P}: x \in \hat{P}\} \subset \{\hat{P}: u \in \hat{P}\}$. But, $x \in I = k(\mathfrak{H})$ implies $\{\hat{P}: u \in \hat{P}\} \subset \{\hat{P}: x \in \hat{P}\}$, hence equality. So, the prime ideals P such that $u \in \hat{P}$ are precisely those for which $x \in P$.

(2) $\Rightarrow$ (3). By (2), the prime ideals P such that $x \in P$ are precisely those for which $y \in P$. By property (A) mentioned above, x and y generate the same principle ideal.

(3) $\Rightarrow$ (1). Let $\mathfrak{F} = \{\hat{P} \in \mathfrak{W}: u \in \hat{P}\}$ be a basic closed set in $\mathfrak{W}$ and $\mathfrak{H} = f^{-1}(\mathfrak{F})$. Let $\tilde{I}$ be the ideal $k(\mathfrak{F})$ and $I = k(\mathfrak{H})$. Clearly, then, $\tilde{I} \cap E = I$. We will show that $\tilde{I} = \hat{I}$. Let $v \in \tilde{I}$. Then, $v \in \hat{P}$, all $\hat{P} \in \mathfrak{F}$. By condition (3), there is $x \in E$ such that for all $y \in E$, $u \leqq y \leqq x$ we have y and x generate the same principal ideal. Let $\hat{P} \in \mathfrak{F}$. Then, there is $z \in E$ such that $v \leqq z$ and $z \in P$. But then, $z \wedge x \in P$ and $z \wedge x \leqq x$. So, they generate the same principal ideal, hence $x \in P$. So, $x \in P$ for all $\hat{P} \in \mathfrak{F} \Rightarrow x \in I \Rightarrow$ $\Rightarrow v \in \hat{I}$.

Now then, $\hat{I} = \tilde{I}$ implies that a given prime ideal P in E contains I if and only if $\hat{P}$ contains $\tilde{I}$. But, $\mathfrak{F}$ being closed, the latter occurs only when $\hat{P} \in \mathfrak{F}$ ie., only for $P \in \mathfrak{H}$. So, $P \supset I = k(\mathfrak{H})$ implies $P \in \mathfrak{H}$ ie., $h(k(\mathfrak{H})) = \mathfrak{H}$. So, $\mathfrak{H}$ is closed which says that f is a homeomorphism of $\mathfrak{V}$ and $\mathfrak{W} = f(\mathfrak{V})$.

Using condition (3) of the above theorem an example will be given of a vector lattice E having the Riesz property but such that the mapping f is not a homeomorphism of all pairs $\mathfrak{V}$, $f(\mathfrak{V})$ of structure spaces for E and $\hat{E}$ (respectively).

Example: Let E be the collection of all bounded real-valued functions f on the positive real axis such that there is a t (dependent upon the choice of f) for which f is finite valued on $[t, \infty)$. Under the usual pointwise operations, E is easily shown to be a vector lattice. Since $1 \in E$, the characterization of closed ideals in E is accomplished exactly as for the space S in section 1, ie., I is a closed ideal in E if and only if there is a set $A \subset E$ such that $I = \{f \in E: f(t) = 0, \text{ all } t \in A\}$ $(A = \{t \in [0, \infty): f(t) = 0$, all $f \in I)$. If I is a closed ideal then and $f \in E$, its component functions in I and $I^{\perp}$ are $f\chi_A$ and $f\chi_{A'}$ both of which assume at most one more value than f. Hence, $f\chi_A$ and $f\chi_{A'}$ are elements of E. Thus, E has the Riesz property.

Using the fact that E contains the identically 1 function and all its components, routine verification of the definition tells us that the Dedekind completion $\hat{E}$ of E is the set of all bounded functions on $[0, \infty)$.

We now show that E does not have property (3) of the last theorem. Let $u(t) = \sin^2 t$, $t \geqq 0$. Then, the function u is a positive element of $\hat{E}$. For any function $x \in E$, $x \geqq u$, there exists a natural number n such that x is finite valued in $[n\pi, \infty)$. Let the function y be defined as follows:

$$y(t) = \begin{cases} x(t): & t \notin [n\pi, (n+1)\pi] \\ \sin^2 t: & t \in [n\pi, (n+1)\pi] \end{cases}$$

Then, $y \in E$, $u<y<x$, but there does not exist an integer m such that $my \geqq x$. For in the open interval $J=((n+1/2)\pi, (n+1)\pi)$, there is a $c>0$ such that $x(t) \geqq c$, $t \in J$. However, $\lim y(t)=0$ for $t \to (n+1)\pi$. E, then, does not satisfy the conditions of Theorem 2.5.

The above example verifies "useful result" 1) mentioned before Theorem 2.5. 2) is verified by noting that the space of all sequences with finite range does satisfy condition (3) of the Theorem, but this space is not complete.

A much simpler proof of the homeomorphism of corresponding structure spaces of E and $\hat{E}$ could be obtained had we assumed the more aesthetically pleasing condition that every ideal $\tilde{I}$ in $\hat{E}$ be of the form $\hat{I}$ for some ideal $I \subset E$.

We will show in the next theorem and the example following that this condition is strictly stronger than those of the last theorem and yet not strong enough to imply completeness.

THEOREM 2.6. *The following are equivalent*:

(1) *Every ideal in* $\hat{E}$ *is of the form* $\hat{I}$ *for some ideal* $I \subset E$.

(2) *For each* u *in* $\hat{E}$, $u \geqq 0$ *there is an* $x \in E$ *and a real number* α *such that* $\alpha x \leqq u \leqq x$.

PROOF. (1) $\Rightarrow$ (2). Let $u \in \hat{E}$, $u \geqq 0$. Then, let $I(u)$ be the ideal in $\hat{E}$ generated by u. Then let $I=(I(u)) \cap E$. By the hypothesis, $I(u)=\hat{I}$. So, there exists $x \in I$ such that $u \leqq x$. However, $x \in I$ implies $x \in I(u)$. Hence there is a natural number m for which $x \leqq mu$. But then $1/m\ x \leqq u \leqq x$.

(2) $\Rightarrow$ (1). Let $\tilde{I}$ be an ideal in $\hat{E}$ and $I=\tilde{I} \cap E$. If u is a positive element in $\tilde{I}$, we can choose $x \in E$ and $\alpha>0$ such that $\alpha x \leqq u \leqq x$. Now $\alpha x \leqq u$ implies $x \in I$. But then $u \leqq x$ implies that $u \in \hat{I}$. So, $\tilde{I} \subset \hat{I}$. Since $\hat{I} \subset \tilde{I}$ is clear we have $\hat{I}=\tilde{I}$.

The existence of x such that $\alpha x \leqq u \leqq x$ implies that for any $y \in E$ such that $u \leqq y \leqq x$, we have $\alpha x \leqq y \leqq x$, that is, x and y generate the same principal ideal in E. The conditions of this last theorem, then, imply the conditions of Theorem 2.5. We have noted before that the space of bounded real-valued sequences with finite range satisfies the conditions of 2.5. Since its Dedekind completion is the space m of bounded sequences, picking $u=\{1/n\}$, it is clear that condition (2) of 2.6 cannot be satisfied. The conditions of the latter theorem are thus strictly stronger than those of the former. As mentioned before, however, even these conditions are not strong enough to imply completeness. The following example will illustrate this fact.

Example. Let E be the set of bounded sequences (a_n) of real numbers such that the set of points $\{a_n : n=1, 2, \ldots\}$ has a countable (possibly finite) closure. E is an Archimedean vector lattice with the usual pointwise operations. E is not Dedekind complete since if $\{r_n: n=1, 2, \ldots\}$ is an ordering of all rationals in the closed interval $[0, 1]$ and (a_n) is the sequence which takes the value r_n at n and 0 elsewhere, then the set $\{(a_n): n=1, 2, \ldots\}$ is bounded in E but does not have a supremum in E. It is clear that $\hat{E}=m$, the space of bounded sequences of real numbers. Let $u \in \hat{E}$ be such that $u \geqq 0$. Let u_n be the n^{th} value of the sequence u. We assume without loss of generality that $\sup u_n \leqq 1$. We define a sequence $x=(x_n)$ as follows: $x_n = 1/2^k$ for all n such that $1/2^{k-1} < u_n \leqq 1/2^k$, $k=1, 2, \ldots$. Then, $x \in E$ since 0 is the only accumulation point of the range of x. Moreover, $\frac{1}{2}x \leqq u \leqq x$. This completes the example.

In this example, we have shown that for each positive u in $\hat{E}$, there is $x \in E$ such that $\frac{1}{2}x \leqq u \leqq x$. There is nothing special about the value 1/2. In fact, by giving the sequence x values α^n instead of $1/2^n$, we can obtain x and α such that $\alpha x \leqq u \leqq x$ for any $0<\alpha<1$. The fact that will be of value is that we can find numbers α arbitrarily close to 1, and elements x such that $\alpha x \leqq u \leqq x$. We will add this to the above property (3) in Theorem 2.6 and use the new property to obtain a result concerning dual spaces.

We denote by $\Omega(E)$, the set of all linear functionals on a vector lattice E which are bounded on order intervals of E. $\Omega(E)$ will be the subspace of $\Omega(E)$ of all *order continuous* linear functionals (i.e., $a_\alpha \downarrow 0$ implies $|f|(a_\alpha) \downarrow 0$). Nakano and others have studied both of these spaces extensively. In general, there is no relation between the spaces $\Omega(E)$ and $\Omega(\hat{E})$. However, it has been shown ([3], [4]) that for any Archimedean space E, $\tilde{\Omega}(E)$ and $\tilde{\Omega}(\hat{E})$ are isomorphic.

We are able here to show that $\Omega(E)$ is isomorphic to $\Omega(\hat{E})$ for a certain class of vector lattices E, namely those with the "new property" mentioned in the last paragraph. For, this property allows us to prove the central result needed to get the isomorphism of $\tilde{\Omega}(E)$ and $\tilde{\Omega}(\hat{E})$, which is: For $u \in \hat{E}$ and $f \in \tilde{\Omega}(E)$, $f \geqq 0$,

$$\sup\,[f(x): x \leqq u,\ x \in E] = \inf\,[f(y): y \geqq u: y \in E].$$

THEOREM 2.7. *If for each positive $u \in \hat{E}$ and $\varepsilon>0$, there is an $x \in E$ and positive scalar α such that $|1-\alpha|<\varepsilon$ and $\alpha x \leqq u \leqq x$, then $\Omega(E)$ is isomorphic to $\Omega(\hat{E})$.*

PROOF. We show first that if $f \in \Omega(E)$, then for $u \in \hat{E}^+$,

$$\sup\,[f(x): x \leqq u,\ x \in E] = \inf\,[f(y): y \geqq u,\ y \in E].$$

Let $p = \sup\,[f(x): x \leqq u,\ x \in E]$, $q = \inf\,[f(y): y \geqq u,\ y \in E]$ and $\varepsilon>0$. If α and x_α are chosen so that $1-\alpha<\frac{1}{2}$ and $\alpha x_\alpha \leqq u \leqq x_\alpha$, then $\alpha>\frac{1}{2}$ implies $x_\alpha \leqq 1/\alpha u \leqq 2u$, hence $f(x_\alpha) \leqq 2f(u)$. Choose any fixed $x_0 \in E$ such that $u \leqq x_0$, $f(x_0)>0$ and let $\delta = \min\,(1/2, \varepsilon/2f(x_0))$. Then, by hypothesis, we can find

β and x_β such that $1-\beta \leqq \delta$ and $\beta x_\beta \leqq u \leqq x_\beta$. But then, $q-p \leqq f(x_\beta) - f(\beta x_\beta) = (1-\beta)f(x_\beta) \leqq 2(1-\beta)f(u) \leqq 2(1-\beta)f(x_0) < \varepsilon$. Since ε was chosen arbitrarily, we have that $p=q$. (The remainder of the proof is similar to that in [4] for $\tilde{\Omega}(E)$).

Given $f \in \Omega(E)$, $f \geqq 0$, define a functional $\hat{f}$ on $\hat{E}^+$ as follows: For $u \in \hat{E}^+$, let $\hat{f}(u) = \sup\,[f(x): x \leqq u,\, x \in E]$. We show first that $\hat{f}$ is additive on $\hat{E}^+$. Let u, v be in $\hat{E}^+$. Then

$$\begin{aligned}
\hat{f}(u+v) &= \sup\,[f(z): z \leqq u+v,\, z \in E] \\
&\geqq \sup\,[f(x+y): x \leqq u,\, y \leqq v,\, x \in E,\, y \in E] \\
&= \sup\,[f(x): x \leqq u,\, x \in E] + \sup\,[f(y): y \geqq u,\, y \in E] \\
&= \hat{f}(u) + \hat{f}(v).
\end{aligned}$$

From above, we have also that

$$\begin{aligned}
\hat{f}(u+v) &= \inf\,[f(w): w \geqq u+v,\, w \in E] \\
&\leqq \inf\,[f(s+t): s \geqq u,\, t \geqq v,\, s \in E,\, t \in E] \\
&= \inf\,[f(s): s \geqq u,\, s \in E] + \inf\,[f(t): t \geqq v,\, t \in E] \\
&= \hat{f}(u) + \hat{f}(v).
\end{aligned}$$

Hence, $\hat{f}(u+v) = \hat{f}(u) + \hat{f}(v)$. Positive homogeneity is clear.

We extend $\hat{f}$ to all of $\hat{E}$ by defining for $u \in \hat{E}, \hat{f}(u) = \hat{f}(u^+) - \hat{f}(u^-)$. It is easily shown then that $\hat{f}$ is a bounded linear functional on $\hat{E}$. Routine verification demonstrates that the mapping $f \to \hat{f}$ from the positive cone of $\Omega(E)$ to that of $\Omega(\hat{E})$ generates an order isomorphism of $\Omega(E)$ onto $\Omega(\hat{E})$.

3. *Discrete Vector Lattices*

If the vector lattice E does not satisfy the Riesz property, the general comparisons of structure spaces for E and $\hat{E}$ cannot be made. For example, if c is the space of convergent sequences and c_0 is the space of zero-convergent sequences, c_0 is a prime ideal in c. However, in m, which is the Dedekind completion of c, the ideal c_0 is not prime. Thus, the mapping f does not even take prime ideals into prime ideals. We notice, however, that in the case of the vector lattice c, there is at least one homeomorphism between a structure space for c and its correspondent in m, and this one is a discrete topological space. We will show that this is the case for a class of spaces which Nakano calls discrete vector lattices. To this end, we first establish two general theorems.

Let E be an arbitrary vector lattice.

THEOREM 3.1. *Let $\mathfrak{P}$ be a structure space for E. Then, the following are equivalent*:

(1) *$\mathfrak{P}$ is a discrete topological space.*

(2) *$\mathfrak{P}$ is a set of closed prime ideals in E.*

(3) *$\mathfrak{P} \sim \{P\}$ is not a structure space for any $P \in \mathfrak{P}$.*

PROOF. (1) $\Rightarrow$ (2). For any P in $\mathfrak{B}$, P is an isolated point and hence open in $\mathfrak{B}$. By [1] (Proposition 2.2), $k(\{P\})$ is a closed ideal. But, $k(\{P\})=P$ and hence the latter is closed.

(2) $\Rightarrow$ (3). Suppose that $\mathfrak{B} \sim \{P\}$ is a structure space for some P in $\mathfrak{B}$. P is a closed prime ideal, hence $P^{\perp} \neq \{0\}$. However, $P \cap P^{\perp}=\{0\}$ and hence for each Q in $\mathfrak{B} \sim \{P\}$ either $P \subset Q$ or $P^{\perp} \subset Q$. Now $P \not\subset Q$ for any such Q since closed prime ideals are necessarily minimal. So, $P^{\perp} \subset Q$ for all Q in $\mathfrak{B} \sim \{P\}$. But this contradicts the assumption that $\mathfrak{B} \sim \{P\}$ is a structure space.

(3) $\Rightarrow$ (1). Suppose $P \in \mathfrak{B}$. Then, by (3), we have that $I= \cap \{Q\colon Q \in \mathfrak{B}, Q \neq P\} \neq \{0\}$, which implies that $I \not\subset P$. Hence, $P \notin h(k(\mathfrak{B} \sim \{P\}))$. The latter set must then equal $\mathfrak{B} \sim \{P\}$, and it follows that $\mathfrak{B} \sim \{P\}$ is closed, hence $\{P\}$ is open.

Note: (3) implies that $\mathfrak{B}$ actually consists of all closed prime ideals in E.

Let E be a vector lattice. A positive element p of E is a *discrete element* of E if $0<|x| \leqq p$ for x in E implies the existence of a scalar c such that $cx=p$. A set $\{a_\alpha\colon \alpha \in \mathfrak{A}\}$ of positive elements *generates* a vector lattice E if the set is orthogonal and $a_\alpha \wedge x=0$ for all $\alpha \in \mathfrak{A}$ implies that $x=0$. A discrete vector lattice is one with a generating set of discrete elements. (This differs from the definition given in [6] in that we do not assume that E is Dedekind complete.) c is an example of a discrete vector lattice.

THEOREM 3.2 *Let E be a discrete vector lattice. Then, E has a structure space which is a discrete topological space.*

PROOF. Let $\{p_\lambda\colon \lambda \in \Lambda\}$ be a generating set of discrete elements. For each λ, let $M_\lambda=\{x \in E\colon |x| \wedge p_\lambda=0\}$. Then $\mathfrak{B}=\{M_\lambda\colon \lambda \in \Lambda\}$ is a structure space for E and $\cap \{M_\lambda\colon \lambda \neq \lambda_0\} \neq \{0\}$ for each $\lambda_0 \in \Lambda$. By Theorem 3.1, then, $\mathfrak{B}$ is a discrete topological space.

The next theorem will give us the homeomorphism mentioned in the introduction to section 3.

THEOREM 3.3. *Let E be an Archimedean vector lattice. Then E is a discrete vector lattice if and only if $\hat{E}$ is a discrete vector lattice. Moreover, the cardinality of the generating sets is the same.*

PROOF. If $\{p_\lambda\colon \lambda \in \Lambda\}$ is a generating set of discrete elements for E, then, from the fact that elements of $\hat{E}$ are majorized by elements of E and vice versa, it is easily shown that $\{p_\lambda\colon \lambda \in \Lambda\}$ is a generating set of discrete elements for $\hat{E}$, once we have shown that each p_λ is discrete in $\hat{E}$. If $u \in \hat{E}$, $u>0$, and $u \leqq p_\lambda$ then there is a net $\{x_\alpha\} \subset E$ such that $x_\alpha \uparrow u$. For each α, there is a scalar c_α such that $c_\alpha x_\alpha=p_\lambda$. ie., $1/c_\alpha p_\lambda=x_\alpha$. But then, if $\beta_\alpha=1/c_\alpha$, we have $\beta_\alpha p_\lambda \uparrow u$, hence $\{\beta_\alpha\}$ is a Cauchy net of scalars. So, $\{\beta_\alpha\}$ converges to β and it follows that $\beta p_\lambda=u$. Hence, $1/\beta u=p_\lambda$. The result is extended additively to arbitrary $u \in E$.

Suppose $\{\hat{p}_\lambda : \lambda \in \Lambda\}$ is a generating set of discrete elements for $\hat{E}$. Then for each λ, $\hat{p}_\lambda \in E$, and $\{\hat{p}_\lambda : \lambda \in \Lambda\}$ is clearly a generating set of discrete elements for E. That the cardinality is the same is easily shown.

COROLLARY 3.4. *If E is an Archimedean discrete vector lattice, then f is a homeomorphism of the discrete topological structure spaces of all closed prime ideals in E with those in $\hat{E}$.*

PROOF. Let $\{p_\lambda : \lambda \in \Lambda\}$ be a generating set of discrete elements of E. Then, from the proof of Theorem 3.3, this set is also a generating set of discrete elements for $\hat{E}$. So, let $M_\lambda \subset E$ be defined as in Theorem 3.2. Then, the corresponding set $\{u \in \hat{E} : |u| \wedge p_\lambda = 0\}$ is equal to $\hat{M}_\lambda$. Let $\mathfrak{B} = \{M_\lambda : \lambda \in \Lambda\}$ and $\mathfrak{W} = \{\hat{M}_\lambda : \lambda \in \Lambda\}$. By Theorems 3.2 and 3.1, then, they are respectively the set of all closed prime ideals in E and $\hat{E}$. Both are discrete topological spaces and since they have the same cardinality, f is a homeomorphism.

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